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Spinors and scalars on Riemann surfaces

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Abstract. We study (magnetic) Dirac and Schrödinger operators for general two-dimensional manifolds. We find relations between them, and use these as *tools* for calculating ground-state degeneracies of various electronic systems (with and without spin). We find examples which have a degenerate ground state, although there are no symmetry operators commuting with the Schrödinger operator. The connection between the two operators is related to a Berry phase.

1. Introduction

When one considers Schrödinger operators on Riemannian manifolds with constant curvature in the presence of a constant magnetic field perpendicular to the surface, one finds degenerate Landau levels. The degeneracies are computable, and so are several of the magneto-transport properties ([1, 2] and references therein).

Remmembering that degeneracy of energy levels is very unusual for Schrödinger operators, a natural question to ask is 'Can we find other (non-constant curvature) surfaces which give degenerate energy levels?'.

If we restrict ourselves to the *ground-state degeneracy*, it turns out that an easy way to do this is by using the Atiyah–Singer index theorem for the *Dirac* operator (as will be explained soon).

A Dirac operator is a first-order differential operator that describes a (relativistic) spin-half particle. The number of its zero modes is very stable against perturbations (a more precise statement appears in the next section). In particular, for a two-dimensional system in a perpendicular magnetic field, the number of zero modes is generically of the order of the number of magnetic flux quanta through the surface.

When we write the Dirac operator explicitly, we find that its square resembles the Schrödinger operator. This connection enables us to calculate the ground-state degeneracy of the Schrödinger operator, using the index theorem for Dirac operators.

We emphasize that although we are using the Dirac operator as a *mathematical tool*, it is indeed relevant for the description of electrons (particles with spin) on curved manifolds. This can be shown by deriving the appropriate Schrödinger–Pauli operator—which turns out to be proportional to the square of the Dirac operator.

The paper is organized as follows. In section 2 we present both the Dirac and Schrödinger operators for a particle on an arbitrary two-dimensional surface, under the action of arbitrary magnetic fields. In section 3 we use these to get the ground-state degeneracy for various electronic systems (with and without spin). In section 4 we give a 'physical explanation' for the effective magnetic field we got, as a manifestation of a Berry phase in real space.

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2. Dirac and Schrödinger operators

We present the Dirac and Schrödinger operators. We give explicit expressions only for a particle on a two-dimensional surface, (which is what we need). Although this result is not new, we give the Dirac operator in a form much simpler than usually found in textbooks.

For convenience, we chose to work with the system of units where $\hbar = c = 2m_e = 1$, and we absorb the electron's charge in the definition of the magnetic field.

We write the Dirac equation in the form $E\psi = \left(\not{D} + \beta m\right)\psi$, where $\not{D}$ denotes the Dirac operator on the surface, and β is a constant Hermitian matrix satisfying $\beta^2 = I$. There is a prescription for writing $\not{D}$, with the 'spin-connection' formalism (see, for example, [5]). When simplified, it gives, for a free particle on a two-dimensional surface, with a metric tensor $g_{\mu\nu}$:

$$\not{D} = -\mathrm{i}\Sigma^a\left(E_a^\mu \partial_\mu + \tfrac{1}{2}E_a^\mu \partial_\mu \ln\sqrt{g} + \tfrac{1}{2}\partial_\mu E_a^\mu\right). \tag{1}$$

Here Σ^a denotes the Hermitian Dirac matrices, satisfying

$$\{\Sigma^a, \Sigma^b\} = 0 \quad (a \neq b) \qquad \{\Sigma^a, \beta\} = 0 \qquad (\Sigma^a)^2 = I \tag{2}$$

(curly brackets stand for anti-commutators). g denotes the determinant of the metric tensor, and E_a^μ are numbers which satisfy:

$$\delta^{ab} E_a^\mu E_b^\nu = g^{\mu\nu} \qquad g_{\mu\nu} E_a^\mu E_b^\nu = \delta_{ab} \tag{3}$$

Note that $\not{D} = \Sigma^a\{E_a^\mu P_\mu\}_{\text{h.p.}}$, where $P_\mu \equiv -\mathrm{i}\partial_\mu$ is the usual (flat) momentum operator, and $\{E_a^\mu P_\mu\}_{\text{h.p.}}$ denotes the Hermitian part of the operator (the hermeticity is relative to the metric).

To introduce a magnetic field $\boldsymbol{B}$, all we have to do is replace the ordinary derivative by a covariant one $-\mathrm{i}\partial_\mu \rightarrow -\mathrm{i}\partial_\mu - A_\mu$, where $\mathrm{d}\boldsymbol{A} = \boldsymbol{B}$.

We see from these definitions that for a given surface, we have many degrees of freedom in writing the Dirac equation. First, we can choose a coordinate system (for example, on the plane we can use Cartesian coordinates, polar coordinates etc). This fixes the metric tensor. Then, we have to choose E_a^μ, since if E_a^μ satisfies the conditions of equation 3, so does $O_b^a E_a^\mu$, where O_b^a is any orthogonal matrix. Next, we have the gauge freedom of the magnetic vector potential A, and finally, we have to choose our (constant) Σ matrices.

Naturally, we choose the easiest coordinate system to work with, and take *a conformal metric tensor* (in two dimensions this can always be done):

$$g_{\mu\nu} = \mathrm{e}^{2\sigma}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

where σ is a function of the coordinates.

We also choose:

$$E_a^\mu = \mathrm{e}^{-\sigma}\delta_a^\mu \tag{5}$$

(Note that it is not always possible to choose one coordinate system which covers the entire surface. In such cases, one covers the manifold by a few patches with a definite coordinate system in each, and gives 'transition functions' among them.)

We choose our Dirac matrices to be $\Sigma^a = \sigma^a$, where σ^a are Pauli sigma matrices, and a is either 1 or 2.

The Dirac operator now reads

$$\begin{aligned}\not{D} &= -\mathrm{i}\mathrm{e}^{-\sigma}\left(\sigma^1(D_1 + \tfrac{1}{2}\partial_1\sigma) + \sigma^2(D_2 + \tfrac{1}{2}\partial_2\sigma)\right) \\ &= -\mathrm{i}\mathrm{e}^{-\sigma}\begin{pmatrix} 0 & 2\partial_z + \partial_z\sigma - 2\mathrm{i}\bar{a} \\ 2\partial_{\bar{z}} + \partial_{\bar{z}}\sigma - 2\mathrm{i}a & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & K^\dagger \\ K & 0 \end{pmatrix}\end{aligned} \tag{6}$$

where we use the notations: $z \equiv x_1 + ix_2$, $\bar{z} \equiv x_1 - ix_2$, $a \equiv \frac{1}{2}(A_1 + iA_2)$, $\bar{a} \equiv \frac{1}{2}(A_1 - iA_2)$. Taking $\beta = \sigma^3$, we get the Dirac equation,

$$\begin{pmatrix} m & K^\dagger \\ K & -m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{7}$$

(Remember that E is the *total* energy, expressing the contribution of both the kinetic energy E_k and the mass.)

The Schrödinger operator for an electron on the same surface, under the action of a magnetic field $B' = dA'$, $a' \equiv \frac{1}{2}(A'_1 + iA'_2)$ is

$$\begin{aligned} H_s(B') &= -e^{-2\sigma} \, [\, (\partial_1 - iA'_1)^2 + (\partial_2 - iA'_2)^2 \,] \\ &= -e^{-2\sigma} \, [\, 4\partial_{z\bar{z}} - 4i(a'\partial_z + \bar{a}'\partial_{\bar{z}}) - 2i(\partial_z a' + \partial_{\bar{z}}\bar{a}') - 4a'\bar{a}']. \end{aligned} \tag{8}$$

To see the connection between the Dirac and Schrödinger operators, notice that:

$$\not{D}^2 = \begin{pmatrix} K^\dagger K & 0 \\ 0 & KK^\dagger \end{pmatrix} \tag{9}$$

where

$$\begin{aligned} K^\dagger K(B) = -e^{-2\sigma} [& 4\partial_{z\bar{z}} - (\partial_z\sigma)(\partial_{\bar{z}}\sigma) + 2(\partial_{z\bar{z}}\sigma) \\ & \times 2((\partial_{\bar{z}}\sigma)\partial_z - (\partial_z\sigma)\partial_{\bar{z}}) + 2i(a(\partial_z\sigma) - \bar{a}(\partial_{\bar{z}}\sigma)) \\ & -4i(a\partial_z + \bar{a}\partial_{\bar{z}}) - 4i(\partial_z a) - 4a\bar{a}]. \end{aligned} \tag{10}$$

Comparing this with the Schrödinger operator, H_s, and choosing $a' = a + \frac{i}{2}(\partial_{\bar{z}}\sigma)$, we get

$$K^\dagger K(B) = H_s(B - \tfrac{1}{2}k) - (B - \tfrac{1}{2}k). \tag{11}$$

Note that if a is a vector potential for a magnetic field B (not necessarily constant), then a' is a vector potential for a magnetic field $B - \frac{1}{2}k$, where k is the Gaussian curvature of the surface, $k = -4e^{-2\sigma}\partial_{z\bar{z}}\sigma$.

Similarly, we get:

$$KK^\dagger(B) = H_s\left(B + \frac{k}{2}\right) + \left(B + \frac{k}{2}\right). \tag{12}$$

To summarize our results: we have found that

$$\not{D}^2(B) = \begin{pmatrix} H_s(B - \frac{1}{2}k) - (B - \frac{1}{2}k) & 0 \\ 0 & H_s(B + \frac{1}{2}k) + (B + \frac{1}{2}k) \end{pmatrix}. \tag{13}$$

We give two examples. On a plane, $k = 0$ and the operator reads

$$\not{D}^2(B) = \begin{pmatrix} H_s(B) - B & 0 \\ 0 & H_s(B) + B \end{pmatrix}. \tag{14}$$

On a general surface, in the absence of a magnetic field, we get

$$\not{D}^2(B = 0) = \begin{pmatrix} H_s(B = -\frac{k}{2}) + \frac{1}{2}k & 0 \\ 0 & H_s(B = k\frac{1}{2}) + \frac{1}{2}k \end{pmatrix}. \tag{15}$$

3. Degeneracies of Energy Levels

We demonstrate how to calculate the ground state, and the degeneracies of other levels, for certain Schrödinger operators.

3.1. Ground-state degeneracies of Schrödinger operators

We explicitly calculate the ground-state degeneracy of a Schrödinger operator describing an electron on an *arbitrary* closed Riemann surface, under the action of a *constant* magnetic field.

Let us start with a known result: for a compact, closed, two-dimensional surface,

$$\text{Index } \not{D} \equiv \text{Dim Ker } K - \text{Dim Ker } K^{\dagger} = \frac{1}{2\pi} \int B \tag{16}$$

where Dim Ker $K(K^{\dagger})$ denotes the dimension of the function space for which $Kf = 0$ ($K^{\dagger} f = 0$). This is a special case of the Atiya–Singer index theorem [3]. Notice that the index equals the ground-state degeneracy of the operator $\not{D}^2$ if and only if one of the kernels is empty.

We use this equation to calculate the ground-state degeneracy of a *Schrödinger* operator for a particle on such a surface:

$$\begin{aligned} H_s(B) &= K^{\dagger}K(B + \tfrac{1}{2}k) + B = KK^{\dagger}(B - \tfrac{1}{2}k) - B \\ H_s(B + k) &= K^{\dagger}K\left(B + \frac{3k}{2}\right) + (B + k) = KK^{\dagger}(B + \tfrac{1}{2}k) - (B + k). \end{aligned} \tag{17}$$

Assume B is constant (otherwise there in no reason for the ground state to be degenerate).

If we choose B and $B + k$ to be everywhere positive, we immediately find that Dim Ker $K^{\dagger}(B - \frac{1}{2}k) =$ Dim Ker $K^{\dagger}(B + \frac{1}{2}k) = 0$ (as H_s is positive definite), hence in this case, the ground-state degeneracy of $H_s(B)$, $d_0[H_s(B)]$, is:

$$\begin{aligned} d_0[H_s(B)] &= \text{Dim Ker } K(B + \tfrac{1}{2}k) = \text{Index}[\not{D}(B + \tfrac{1}{2}k)] \\ &= \frac{1}{2\pi} \int (B + \tfrac{1}{2}k) = \frac{1}{2\pi} \int B + (1 - g) \end{aligned} \tag{18}$$

where g is the genus of the surface (the number of handles: $g = 0$ for a sphere, $g = 1$ for a torus etc). For the case where B and $B - k$ are negative, we have to replace B by $|B| = -B$.

We get the following *general result*: the ground-state degeneracy for a Schrödinger operator describing a spinless particle on a compact, closed manifold, under the action of a constant magnetic field B perpendicular to the surface, is the number of magnetic flux quanta through the surface plus $1 - g$, provided the magnetic field is strong enough ($B, B + k > 0$ everywhere). If it is not, it is only a *lower bound* on the degeneracy. We make the following remarks:

(a) The ground-state degeneracy contains information about the *topology of the manifold* (contrary to the spinor case where only the total magnetic flux determines the degeneracy). Moreover, the degeneracy does not follow from any *symmetry* principle, and we know of no reason for other energy levels to be degenerate.

(b) Contrary to the case of a the Dirac operator, for a Schrödinger operator a degeneracy would exist only if the magnetic field is *constant*. Any deviation from this situation can remove it.

(c) We found the ground-state degeneracy for *any* compact, closed surface. For example, *this shows that for all tori, the ground-state degeneracy is equal to the number of* magnetic flux quanta through the surface, independent of the specific shape of the torus, as long as the magnetic field is constant. This result was known by explicit calculations only for the flat torus.Note, however, that while this result holds unconditionally for the flat torus, here we have conditions on the strength of the magnetic field.

(d) In [8], Wen and Zee introduced the notion of a *shift*, which is the difference between the number of flux quanta through the surface and the ground-state degeneracy. We proved that the value of this *shift* for a Schrödinger particle is $(1-g)$. We remark that for a Dirac particle, under similar but *local* conditions on B, the shift vanishes. (These conditions assure that only one spin state contributes to the index, and hence its absolute value equals the number of zero modes.)

3.2. Calculating the energies and degeneracies of Landau levels for constant-curvature surfaces

Generically, only the ground state of a Schrödinger operator will be degenerate if the operator is defined over a *general* manifold. (There is no symmetry operator commuting with the Hamiltonian). It is known, however, that there is a class of surfaces where all the Landau levels are degenerate: those with a constant curvature. We want to demonstrate how to calculate the energy levels and their degeneracies using the Atiya–Singer index theory. We restrict ourselves to compact, smooth surfaces.

We assume that B and k are constant, and $B+\frac{1}{2}k>0$. In this case, $\operatorname{Ker} KK^{\dagger}(B)$ is empty, while $\operatorname{Ker} K^{\dagger}K(B)$ is not. Because $\operatorname{Spec}(K^{\dagger}K/0) = \operatorname{Spec}(KK^{\dagger}/0)$, all other eigenvalues of these operators are the same, including multiplicities. This tells us that the first excited eigenvalue of $K^{\dagger}K$ equals the lowest eigenvalue of $KK^{\dagger}$. But using equations 11 and 12, we find that this eigenvalue is $2B+k$. This gives us the first excited energy level of $H_s(B-\frac{1}{2}k)$, and hence the first excited energy of $H_s(B+\frac{1}{2}k)$, $(H_s(B+\frac{1}{2}k) = H_s(B'-\frac{k}{2}), B' \equiv B+\frac{k}{2})$. from this we derive the second excited eigenvalue of $K^{\dagger}K$, and so on.

Denoting by $\lambda_q^{KK^{\dagger}}$ and $\lambda_q^{K^{\dagger}K}$ the qth eigenvalue of $KK^{\dagger}$ and $K^{\dagger}K$ respectively, and by $E_q(B')$ the qth energy level for the Schrödinger operator with magnetic field B', this procedure gives us

$$E_q(B-\tfrac{1}{2}k) = \lambda_q^{K^{\dagger}K}(B) + (B-\tfrac{1}{2}k)$$
$$E_q(B+\tfrac{1}{2}k) = \lambda_q^{KK^{\dagger}}(B) - (B+\tfrac{1}{2}k). \tag{19}$$

From which we obtain

$$\lambda_q^{KK^{\dagger}}(B) = \lambda_q^{K^{\dagger}K}(B) + (2B+k) = \lambda_{q+1}^{K^{\dagger}K}(B). \tag{20}$$

Using $\lambda_0^{K^{\dagger}K}(B)=0$, we get

$$\lambda_q^{K^{\dagger}K}(B) = 2qB + q^2k \tag{21}$$

and

$$E_q(B) = \lambda_q^{K^{\dagger}K}(B+\tfrac{1}{2}k) + B = (2q+1)B + q(q+1)k. \tag{22}$$

For example, the energy levels for a Schrödinger particle on a flat torus are equally spaced, $E_q = 2qB$. On a sphere the spacing between two consecutive levels increases with energy, while on a negative curvature manifold it decreases.

Calculating the *degeneracies* now becomes simple: the degeneracy of the qth energy level of $H_s(B-\frac{1}{2}k)$ equals the degeneracy of the $(q-1)$th energy level of $H_s(B+\frac{1}{2}k)$. Knowing the ground-state degeneracy $d_0(H_s(B-\frac{1}{2}k)) = \frac{1}{2\pi}\int B$, we get $d_q(H_s(B-\frac{1}{2}k)) = \frac{1}{2\pi}\int B + 2q(1-g)$, or

$$d_q(H_s(B)) = \frac{1}{2\pi}\int B + (2q+1)(1-g) \tag{23}$$

where $d_q(H_s(B))$ denotes the q'th energy level degeneracy. It is easy to verify that for non-negative k equation 22, 23 holds for every $q \in Z_+$, while for negative k it holds only for q's satisfying: $q < (B/|k|) - 1$ (remember that we calculated $d_0 H(B')$ assuming $B', B' + k > 0$).

We note that we can also reverse the direction, and use known results on *Schrödinger* operators, to find the number of zero modes for the Dirac operator. (For example, on non-compact, finite-area, constant negative surface [1]).

4. The effective magnetic field and Berry's phase

We explain why the Dirac equation with a given magnetic field is connected to a Schrödinger equation with a *different* magnetic field, which depends on the geometrical curvature of the surface. The source of this is clear: eigenspinors of the form

$$\begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}$$

correspond to a spinor pointing perpendicular to the manifold, the spin pointing in the direction parallel or anti-parallal to the (local) area form. In our discussion, we always pick one of these forms, depending on the sign of the magnetic field. Therefore, we consider a system for which the direction of the spin is at any point perpendicular to the surface, and because of that, if the curvature does not vanish, this direction is changing (imagine an embedding of the surface into a flat Eucleadian space of higher dimension. For example, a sphere in $\mathbb{R}^3$).

To clarify this point, let us concentrate on the embedding of the surface, and use a *fixed* coordinate system (the Dirac equation then has its usual form). If we choose continuous wave functions on the surface,

$$\psi_{\text{up,down}} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_{\text{up,down}}$$

(the labels refer to the spin direction: *up* if it is in the direction of the area form, and *down* if it is in the opposite direction). Then, generically, because the direction of the area form varies, we have

$$-\mathrm{i}\langle \psi_{\text{up,down}} | d\psi_{\text{up,down}} \rangle \equiv A^{\text{eff}}_{\text{up,down}} \neq 0. \tag{24}$$

The effective field we get is simply $B^{\text{eff}} = \mathbf{d}A^{\text{eff}}$. Note that these wavefunctions cannot be continuously extended to the entire embedding space.

A very clear and simple example can be found in a paper by Stone [7]: when an electron is forced to move while its spin points in the $\hat{r}$ direction (using the usual spherical coordinates of $\mathbb{R}^3$), we get, for $s_r = \pm 1/2$, an effective magnetic field of a monopole with strength $\pm\frac{1}{2}$ at the origin. (Take, for example, the *up* spin state:

$$\psi_{\text{up}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi} \end{pmatrix}$$

$$A^{\text{eff}}_{\text{up}} \equiv -\mathrm{i}\langle \psi_{\text{up}} | d\psi_{\text{up}} \rangle = \tfrac{1}{2}(1 - \cos\theta)\,\mathrm{d}\phi$$

and hence

$$B^{\text{eff}}_{\text{up}} = \tfrac{1}{2}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

and if the spinor is confined to a sphere of radius R, $B^{\text{eff}}_{\text{up}} = \frac{1}{2R^2}\mathrm{d}(\text{area}) = \frac{k}{2}\mathrm{d}\,\text{area})$.

Here we should point out that there is a known, related, general claim that performing the Born–Oppenheimer (BO) approximation for a spin in high magnetic field and then integrating out the spin degree of freedom, leads to an effective magnetic field. In our case, dealing with *intrinsically* two-dimensional systems did, automatically, the job of the BO approximation.

That is, the effective magnetic field we got is simply the curl of a Berry phase [4], emerging from the spin holonomy.

Contrary to Berry's original example, here the field direction depends on two 'real' coordinates, rather than on two parameters, and therefore we get an effective magnetic field in the real space and not in a parameter space.

Interestingly enough, the mixed situation (where the phase depends on one spatial coordinate and one parameter) was also considered: Stern describes in [6] an example of a spinor living on a ring located on the x–y plane, under the action of a magnetic field $\boldsymbol{B} = (\cos\alpha\hat{z} + \sin\alpha\hat{\theta})$. We see that the direction of the field depends both on the parameter α, and the coordinate around the ring, θ. A short calculation gives $A^{\rm eff}_{\rm up,down} = \frac{1}{2}(1 \pm \cos\alpha)\mathrm{d}\theta$, and so, $\oint A^{\rm eff}_{\rm up,down} = \pi(1 \pm \cos\alpha)\mathrm{d}\theta$, and persistent currents would be different for each spin direction. (Because we have *one* spatial coordinate, we can define an effective *flux* in real space, but there is no meaning for a definition of an effective *magnetic field*).

To summarize, the effective shift of the magnetic field comes from the fact that we imposed implicitly the constraint that enforces the spin to point at a direction perpendicular to the surface, and got a holonomy effect that caused a Berry phase in real space.

5. Two-dimensional Dirac spinors and three-dimensional electrons

After introducing the Dirac and Schrödinger operators on two-dimensional surfaces, it is natural to ask if these equations can describe electrons in thin films, in the three dimensional world. In some cases, the answer to this question is positive.

For a Schrödinger particle, if the thin film induces no significant effective potential and the energy scale of its width is high enough, we can neglect the third dimension.

To understand the relevance of the Dirac operator to real experimental set-ups, one has to build an appropriate Schrödinger–Pauli operator for the non-relativistic electrons. To first-order approximation, this operator equals $\not{p}^2$, and the equation describing 'two-dimensional' electrons is $\not{p}^2\psi = \mathrm{i}\partial\psi/\partial t$, where ψ is a two-component wave function. The equation describes 'two-dimensional electrons', and the eigenspinors describe electrons with spin direction *perpendicular* to the surface. In a real experiment, even if we have a two-dimensional electron gas, the spin can point at any direction in space. Therefore, our results are applicable if, and only if, we force the spin to be everywhere perpendicular to the surface. This can be done if we impose sufficiently high magnetic fields perpendicular to the manifold. Note that where the Gaussian curvature is large, this is not a trivial task at all, but it will do as a 'gedanken experiment'.

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